

## AFFINE TANGLES AND IRREDUCIBLE EXOTIC SHEAVES.

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ABSTRACT. We construct a weak representation of the category of framed affine tangles on a disjoint union of triangulated categories  $\mathcal{D}_{2n}$ . The categories we use are that of coherent sheaves on Springer fibers over a nilpotent element of  $\mathfrak{sl}_{2n}$  with two equal Jordan blocks. This representation allows us to enumerate the irreducible objects in the heart of the exotic  $t$ -structure on  $\mathcal{D}_{2n}$  by crossingless matchings of  $2n$  points on a circle. We also describe the algebra of endomorphisms of the direct sum of the irreducible objects.

## 1. INTRODUCTION

In autumn 2006 Paul Seidel, upon learning the first results of [4], suggested that one could assign functors between categories of coherent sheaves on Springer fibers to tangle diagrams built between two circles (instead of between two segments, as usual); that crossingless matchings of points on a circle enumerate irreducible exotic sheaves ([2]), and that the algebra of endomorphisms of the direct sum of irreducible objects is described as a direct sum of blocks that are tensor products of tensor powers of two-dimensional spaces  $\Lambda^0 V \oplus \Lambda^2 V$  and  $\Lambda^1 V$ , where  $V$  is a two-dimensional symplectic vector space. The algebraic structure on it is described in terms similar to that of [8]. In this paper, we prove this conjecture.

Let  $G$  be a semisimple Lie group,  $\mathfrak{g}$  its Lie algebra. In this paper we will mainly consider the case  $\mathfrak{g} = \mathfrak{sl}_n$ . Let  $\mathfrak{h} \subset \mathfrak{g}$  be its Cartan subalgebra, and let  $\mathcal{N} \subset \mathfrak{g}$  be the nilpotent cone. Let  $\mathcal{B}$  be the space of all Borel subalgebras of  $\mathfrak{g}$  (i.e. the space of full flags  $\mathcal{B} \cong G/B$  for some Borel subgroup  $B$  of  $G$ ). Then the variety  $\tilde{\mathfrak{g}} = \{(x, b) | x \in \mathfrak{g}, b \in \mathcal{B}, x \in b\}$  together with its projection  $\tilde{\pi}$  onto  $\mathfrak{g}$  (which is called the Grothendieck simultaneous resolution) provides a resolution of singularities for  $\mathcal{N} \subset \mathfrak{g}$ , i.e. the map  $\tilde{\mathcal{N}} = \tilde{\pi}^{-1}(\mathcal{N}) \rightarrow \mathcal{N}$  is a resolution of singularities. It is called the Springer resolution.

For each nilpotent  $z \in \mathcal{N}$  consider a transversal (Slodowy) slice  $S_z$  of  $\mathcal{N}$ , and two varieties  $B_z = \tilde{\pi}^{-1}(z) \subset \tilde{\pi}^{-1}(S_z) = U_z$ . It is a classical fact that  $U_z$  is symplectic, and  $B_z$  is its compact Lagrangian subvariety. In this paper we will study the triangulated categories  $\mathcal{D}_z = D_{B_z}^b(\text{Coh } U_z)$  of complexes of coherent sheaves on  $U_z$  with all cohomology set-theoretically supported on  $B_z$ , for special values of  $z$ . It is known ([9]) that the category  $D^b(\text{Coh } \tilde{\mathcal{N}})$  has an action of the affine braid group  $ABr_n$ . The geometric construction of this action suggests that it may be defined on  $\mathcal{D}_z$  for every  $z$ .

The braid group  $Br_n$  on  $n$  strands may be viewed as the group of morphisms  $\text{Hom}([n], [n])$  in the category of tangles. One can develop a theory of affine tangles, so that the affine braid group  $ABr_n$  would be the group of morphisms  $\text{Hom}([n], [n])$  in the category of affine tangles. It turns out that for the series  $Z_{2n}, n \in \mathbb{N} \cup \{0\}$ ,

of nilpotents with two  $n \times n$  Jordan blocks, there is an action of the category of affine tangles on the disjoint union of the corresponding categories  $\mathcal{D}_{2n}$ .

For every  $z \in \mathfrak{g} = \mathfrak{sl}_n$  there is a special  $t$ -structure on  $\mathcal{D}_z$ , called the exotic  $t$ -structure ([2]). It is characterized by the properties of the action of the subsemi-groups  $ABr_n^+ \subset ABr_n$  of positive affine braids, and  $ABr_n^- \subset ABr_n$  of negative affine braids:

$$\begin{aligned} (1) \quad & \{\alpha \in D^{\geq 0}\} \iff \{\forall b^+ \in ABr_n^+ \quad R\Gamma(\Psi(b^+)\alpha) \in D^{\geq 0}(Vect)\}; \\ (2) \quad & \{\alpha \in D^{\leq 0}\} \iff \{\forall b^- \in ABr_n^- \quad R\Gamma(\Psi(b^-)\alpha) \in D^{\leq 0}(Vect)\} \end{aligned}$$

where  $\Psi(b)$  is the action of  $ABr_n$ .

It turns out that functors that correspond to the "cup" tangles, which generate two adjacent strands, are  $t$ -exact with respect to these  $t$ -structures (Claim 6). A theorem by Bezrukavnikov and Mirkovic ([3]) implies then that these functors map irreducible objects to irreducible. In particular, it follows that a flat  $(0, 2n)$  tangle  $\alpha$  with no loops corresponds to a functor from  $\mathcal{D}_0 \cong D^b(Vect)$  to  $\mathcal{D}_{2n}$  that maps the 1-dimensional space to an irreducible object  $E_\alpha$  in the heart of the exotic  $t$ -structure on  $\mathcal{D}_{2n}$ . It turns out that every irreducible is isomorphic to some object obtained in this way.

We will view flat  $(0, 2n)$  affine tangles with no loops as crossingless matchings of points on a circle in a plane (cf. Khovanov's crossingless matchings of points on a line in [8]), drawn inside that circle, with the center of the circle cut out of our plane. This graphical presentation corresponds to  $(0, 2n)$  tangles, i.e. to functors  $\mathcal{D}_0 \rightarrow \mathcal{D}_{2n}$ . To compute  $\text{Hom}^*(E_\alpha, E_\beta)$  for two flat affine  $(0, 2n)$  tangles  $\alpha, \beta$ , one takes the composition of the functor  $\Psi(\beta)$  and the right adjoint of the functor  $\Psi(\alpha)$ , which corresponds to a flat  $(2n, 0)$  tangle  $\alpha^\vee$ . The composition  $\alpha^\vee \beta$  is a flat link. The isomorphism class of the functor  $\Psi(\alpha^\vee \beta) : \mathcal{D}_0 \rightarrow \mathcal{D}_0$  is an invariant of the isotopy class of  $\alpha^\vee \beta$ . It turns out (Proposition 6) that for a flat link  $\alpha^\vee \beta$  the corresponding functor is defined not only up to an isomorphism, but up to a canonical isomorphism, hence we can view the functor itself as an invariant of the isotopy class of the link. An isotopy class of a flat link in a plane is the number of components of the link; in a dotted plane, it is a pair of numbers, that of components which enclose the origin, and that of components which do not. To each component we assign a 2-dimensional graded vector space:  $\mathcal{A} = \mathbb{C}[1] \oplus \mathbb{C}[-1]$  to a component that does not enclose the origin, and  $\mathcal{A}_0 = \mathbb{C}^2[0]$  to a component that does. The space of morphisms  $\text{Hom}^*(E_\alpha, E_\beta)$  is the (appropriately shifted) tensor product of graded vector spaces assigned to components (Theorem 2), and the composition law is described in terms of certain product, coproduct, action and coaction maps between tensor products of  $\mathcal{A}, \mathcal{A}_0$  and their copies. For the description of the composition law see Theorem 4 and the subceding argument.

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## 2. PRELIMINARIES.

**2.1. Spherical functors.** For all proofs the reader is referred to [1].

Let  $\mathcal{D}, \mathcal{D}_0$  be triangulated categories,  $S : \mathcal{D}_0 \rightarrow \mathcal{D}$  a functor. Suppose that  $S$  has left and right adjoints  $L(S), R(S)$ . Then there are four natural morphisms of functors:

$$L(S)S \rightarrow id \quad id \rightarrow SL(S) \quad id \rightarrow R(S)S \quad SR(S) \rightarrow id.$$

Using some additional structure (in our particular case it is the triangulated structure on the category of Fourier-Mukai transforms; for details cf. [1]), one can define the twist functor  $T_S$  to be the cone of  $SR(S) \rightarrow id$ , and the dual twist functor  $T'_S$  to be the cone of  $id \rightarrow SL(S)$ . These functors are exact.

Call the functor  $S$  spherical if it satisfies two following conditions:

- (1) the cone of  $id \rightarrow R(S)S$  is an exact autoequivalence of  $\mathcal{D}_0$ . Let us call this functor  $F_S$ .
- (2) the natural map  $R(S) \rightarrow F_S L(S)$  induced by  $R(S) \rightarrow R(S)SL(S)$  is an isomorphism of functors.

**Claim 1.** ([1]) *If  $S$  is spherical, then the twist  $T_S$  is an autoequivalence.*

**Proposition 1.** ([1]) *Let  $S_1 : \mathcal{D}_1 \rightarrow \mathcal{D}$  and  $S_2 : \mathcal{D}_2 \rightarrow \mathcal{D}$  be spherical functors.*

- (1) *If there exists an equivalence of categories  $X : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ , and  $S_1 \simeq S_2 X$ , then  $T_{S_1} \simeq T_{S_2}$ .*
- (2) *If  $Y : \mathcal{D} \rightarrow \mathcal{D}$  is an autoequivalence, then  $Y S_i$  is also a spherical functor.*
- (3)  *$T_{S_1} T_{S_2} \simeq T_{T_{S_1} S_2} T_{S_1} \simeq T_{S_2} T_{T'_{S_2} S_1}$ .*

**Proposition 2.** ([1]) *If  $S$  is a spherical functor, then the following commutation relations hold:*

- (1)  $T'_S S F_S[1] \simeq S \simeq T_S S F'_S[-1];$
- (2)  $F_S L T'_S[1] \simeq L \simeq F'_S L T_S[-1];$
- (3)  $F_S R T'_S[1] \simeq R \simeq F'_S R T_S[-1].$

**2.2. Horja construction.** There is one particular class of spherical functors, which are defined using a construction that appeared first in the work [5] by R.P. Horja.

Let  $i : D \rightarrow X$  be an embedding of a divisor, and  $\pi : D \rightarrow M$  a  $\mathbb{P}^1$ -bundle.

**Claim 2.** ([1]) *The functor  $i_* \pi^* : D^b(\text{Coh } M) \rightarrow D^b(\text{Coh } X)$  is spherical if and only if the intersection index on  $X$  of the divisor  $D$  and the generic fiber of  $\pi$  is  $-2$ .*

**2.3. Braid group action on  $D^b(\tilde{\mathcal{N}})$ .** Let  $\mathfrak{g} = \mathfrak{sl}_n, \mathcal{B}, \mathcal{N}, \tilde{\mathcal{N}}$  be as in the Introduction. In [9] Khovanov and Thomas have proved that the braid group relations hold for some natural collection of spherical twists on  $D^b(\tilde{\mathcal{N}})$ . This collection of functors is constructed as follows.

For  $1 \leq k \leq n$  let  $P_k$  be the space of partial flags with  $k$ -dimensional space omitted. Denote the forgetful projection by  $p_k : \mathcal{B} \rightarrow P_k$ . Then the intermittent space  $T^* P_k \times_{P_k} \mathcal{B}$  (or, equally, the total space of  $p_k^*(T^* P_k)$ ) is a  $\mathbb{P}^1$ -bundle over  $T^* P_k$  and a divisor in  $T^* \mathcal{B}$ . Denote the projection by  $\pi_k : T^* P_k \times_{P_k} \mathcal{B} \rightarrow T^* P_k$  and the embedding by  $i_k : T^* P_k \times_{P_k} \mathcal{B} \rightarrow T^* \mathcal{B}$ . Then the functors

$$(3) \quad (i_k)_* \pi_k^* : D^b(T^* P_k) \rightarrow D^b(T^* \mathcal{B})$$

are spherical by the following lemma:

**Lemma 1.** ([1]) *The functors (3) satisfy the conditions of Claim 2.*

Denote the corresponding spherical twists by  $Tw_n^k$ . The following result is due to Khovanov and Thomas:

**Claim 3.** ([9]) *The functors  $Tw_n^k$  satisfy the relation*

$$(4) \quad Tw_n^k Tw_n^{k+1} Tw_n^k \simeq Tw_n^{k+1} Tw_n^k Tw_n^{k+1}.$$

The flag variety  $\mathcal{B}$  carries a flag of tautological vector bundles  $\mathcal{V}_k$ ; introduce quotient line bundles  $\mathcal{E}_k = \mathcal{V}_k / \mathcal{V}_{k-1}$ . We will use the same notation for their restrictions to  $B_z$  and also for their various pull-backs.

In our further argument we will use functors  $G_n^k = \mathcal{E}_k \otimes (i_k)_* \pi_k^*$  instead of functors (3). A shift by a line bundle does not impose any effect on the twist by part 1 of Proposition 1.

The braid group action above has yet another description.

Let  $\mathfrak{g}^{reg}$  denote the subspace of regular (not necessarily semi-simple) elements, and let  $\tilde{\mathfrak{g}}^{reg}$  be its preimage in  $\tilde{\mathfrak{g}}$ . Then the map  $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$  gives an isomorphism  $\tilde{\mathfrak{g}}^{reg} \simeq \mathfrak{g}^{reg} \times_{\mathfrak{h}/W} \mathfrak{h}$ , whence we see that the Weyl group  $W$  acts on  $\mathfrak{g}^{reg}$  by acting on the second factor. Let  $\Gamma_\alpha \subset \tilde{\mathfrak{g}}^2$  be the closure of the graph of the action of a simple reflection  $s_\alpha$  on  $\tilde{\mathfrak{g}}^{reg}$ , and let  $\Gamma_\alpha$  be the intersection of  $\Gamma_\alpha$  with  $\tilde{\mathcal{N}}^2$ . Denote by  $\mathbf{pr}_i^\alpha$  (resp.  $pr_i^\alpha$ ), where  $i = 1, 2$ , the projections of  $\Gamma_\alpha$  (resp.  $\Gamma_\alpha$ ) on the two factors.

It is well-known that  $H^2(\mathcal{B}) \cong \mathfrak{h}^*$ . The Picard group of  $\mathcal{B}$  is isomorphic to the root lattice  $\Lambda \subset \mathfrak{h}^*$ . For  $\lambda \in \Lambda$  we will denote by  $\mathcal{O}(\lambda)$  the corresponding line bundle on  $\mathcal{B}$  (and, by abuse of notation, the induced line bundle  $f^* \mathcal{O}(\lambda)$  for any  $f : X \rightarrow \mathcal{B}$ ).

Define an affinization  $AW = W \ltimes \Lambda$  (it is the affine Weyl group of the Langlands dual group  ${}^L G$ ). It (weakly) acts on the abelian category of coherent sheaves on  $\tilde{\mathfrak{g}}^{reg}$ . Let  $ABr$  be the affine braid group associated to  $AW$ . For  $w \in AW$  denote by  $\tilde{w}$  the canonical lifting of  $w$  to the affine braid group. Let  $ABr^+ \subset ABr$  denote the semigroup generated by  $\tilde{w}$  for all  $w \in AW$ . We are going to extend the action of  $AW$  to the whole  $\tilde{\mathfrak{g}}$ , passing from the abelian category to its derived category, and to the braid group from the Weyl group.

**Theorem 1.** ([2]) *There exists an (obviously unique) action of  $ABr^+$  on  $D(\tilde{\mathfrak{g}})$  and  $D(\tilde{\mathcal{N}})$  such that for  $\lambda \in \Lambda^+ \subset \Lambda$  we have  $\tilde{\lambda} : \mathcal{F} \rightarrow \mathcal{F}(\lambda)$  and for a simple reflection  $s_\alpha \in W$  we have  $\tilde{s}_\alpha : \mathcal{F} \rightarrow (\mathbf{pr}_1^\alpha)^* (\mathbf{pr}_2^\alpha)_* \mathcal{F}$  (resp.  $\tilde{s}_\alpha : \mathcal{F} \rightarrow (pr_1^\alpha)^* (pr_2^\alpha)_* \mathcal{F}$ ).*

This action will be used in Section 5 to define the exotic  $t$ -structure on the derived categories involved. In order to link it to the rest of the current work, we prove the following

**Proposition 3.** *For  $\mathfrak{g} = \mathfrak{sl}_n$  the action of  $ABr^+$  on  $D(\tilde{\mathcal{N}})$  from Theorem 1 coincides with the action of  $ABr^+$  on  $D(\tilde{\mathcal{N}})$  generated by twists in the functors (3) and tensor multiplications by line bundles  $\mathcal{O}(\lambda)$ ,  $\lambda \in \Lambda^+$ .*

*Proof.* In the topological description of the braid group the simple reflections  $s_i$  correspond to the elementary generators  $t_n^i(1)$ ,  $i = 1, \dots, 2n-1$ . It suffices to check that the spherical twists  $Tw_n^k$  are naturally isomorphic to the functors  $(pr_1^\alpha)^* (pr_2^\alpha)_*$ . It is done in the language of Fourier-Mukai kernels, for example, in [4].

□

**2.4. Springer fibers.** The Grothendieck resolution is finite outside  $\tilde{\mathcal{N}}$ , and on  $\tilde{\mathcal{N}}$  it may be written down as

$$\tilde{\mathcal{N}} \cong \{(z \in \mathcal{N}; (0 = V_0 \subset V_1 \subset \dots \subset V_n = V)) | zV_i \subset V_{i-1}, i = 1, \dots, n\} \rightarrow \mathcal{N}.$$

Thus for every  $z \in \mathcal{N}$  its preimage  $B_z$  is a compact algebraic variety isomorphically projected to a subvariety of the space  $\mathcal{B}$  of full flags. Choose an affine space  $S_z \subset \mathfrak{g}$ , transversal to the  $\mathfrak{g}$ -orbit of  $z$  at  $z$ , and of complimentary dimension. Denote by  $U_z$  the preimage of  $S_z \cap \mathcal{N}$  in  $\tilde{\mathcal{N}}$ . Denote by  $D_k$  the divisor  $T^*P_k \times_{P_k} \mathcal{B} \subset T^*\mathcal{B} \cong \tilde{\mathcal{N}}$ . In terms of (2.4) we have

$$(5) \quad D_k \cong \{(z \in \mathcal{N}; (0 = V_0 \subset V_1 \subset \dots \subset V_n = V)) | \\ zV_i \subset V_{i-1}, i = 1, \dots, n; zV_{k+1} \subset V_{k-1}\}.$$

Then its intersection with  $U_z$  is a divisor in  $U_z$  and a  $\mathbb{P}^1$ -bundle over  $T^*P_{k,z}$ , where

$$P_{k,z} \cong \{(0 = V_0 \subset V_1 \subset \dots \subset \hat{V}_k \subset \dots \subset V_n = V) | \\ zV_i \subset V_{i-1}, i = 1, \dots, k-1, k+2, \dots, n; zV_{k+1} \subset V_{k-1}\}.$$

Thus the braid group action on  $D^b(\text{Coh } \tilde{\mathcal{N}})$  may be restricted to  $\mathcal{D}_z = D_{B_z}^b(U_z)$ .

### 3. TANGLES

Recall that a smooth  $(n, m)$  tangle is a proper, smooth embedding of  $(n + m)/2$  arcs and a finite number of circles into  $\mathbb{R}^2 \times [0, 1]$  such that the boundary points of the arcs map bijectively on the  $n + m$  points  $(1, 0, 0), \dots, (n, 0, 0), (1, 0, 1), \dots, (m, 0, 1)$ , and the arcs are parallel to  $z$  axis at the ends. A  $(0, 0)$  tangle is called a link. Given an  $(n, m)$  tangle  $T$  and an  $(m, p)$  tangle  $U$ , there is a composition tangle  $T \circ U$ , which is the  $(n, p)$  tangle obtained by stacking  $U$  on top of  $T$  and shrinking the  $z$ -direction. This composition is associative only up to isotopy. We say that a tangle is *properly embedded*, if its projection onto the plane  $(x, z)$  has only simple node singularities, and if  $z$ -coordinates of all nodes and critical points (points where  $z$ -derivative is 0) are different. A *tangle diagram* is then an image of a projection of a properly embedded tangle to  $\mathbb{R} \times [0, 1]$ , with information on the vertical order of strands attached to every node.

**3.1. Generators and relations.** Introduce a category  $Tan_{iso}$  (resp.  $Diag_{iso}$ ) whose objects are natural numbers, and morphisms from  $[n]$  to  $[m]$  are isotopy classes of smooth  $(n, m)$  tangles (resp. isotopy classes of tangle diagrams). There is a functor  $Diag_{iso} \rightarrow Tan_{iso}$ , but not in the other direction. Since every tangle is isotopic to some properly embedded tangle, this functor is surjective on morphisms. This, in order to list the morphisms of  $Tan_{iso}$ , it suffices to describe the morphisms of  $Diag_{iso}$  and the equivalence relation on them which identifies morphisms that map to the same morphism in  $Tan_{iso}$ .

**Lemma-Definition 1.** *The category  $Diag_{iso}$  is equivalent to the category  $CDiag$  of combinatorial diagrams, where objects are natural numbers, and morphisms are composable sequences of the following elementary diagrams:*

- "cups"  $g_n^i$ , which generate strands  $i$  and  $i + 1$  in an  $(n - 2, n)$  tangle (we adopt the convention of reading the tangle diagram from bottom to top);
- "caps"  $f_n^i$  that connect strands  $i$  and  $i + 1$  in an  $(n, n - 2)$  tangle;

- "positive crossings"  $t_n^i(1)$  that cross strands  $i$  and  $i+1$  in an  $(n, n)$  tangle with the  $i$ th strand passing over;
- "negative crossings"  $t_n^i(2)$  that cross strands  $i$  and  $i+1$  in an  $(n, n)$  tangle with the  $i$ th strand passing under.

*Proof.* By definition of tangle diagram, the order of  $z$ -coordinates of critical points and nodes cannot change under isotopy. Then any tangle diagram is isotopic to a unique composition of diagrams that each contain only one node or critical point, and these are precisely the elementary diagrams from the above list.

□.

The equivalence relation on them is described by the following lemma ([7], Lemma X.3; cf. [4]):

**Lemma 2.** *Every isotopy of tangles is a composition of the following elementary isotopies up to isotopies of tangle diagrams:*

- (6) Reidemeister (0) :  $f_n^i \circ g_n^{i+1} \sim id \sim f_n^{i+1} \circ g_n^i$ ;
- (7) Reidemeister (I) :  $f_n^i \circ t_n^{i+1}(2) \circ g_n^i \sim id \sim f_n^i \circ t_n^{i+1}(1) \circ g_n^i$ ;
- (8) Reidemeister (II) :  $t_n^i(2) \circ t_n^i(1) \sim id \sim t_n^i(1) \circ t_n^i(2)$ ;
- (9) Reidemeister (III) :  $t_n^i(1) \circ t_n^{i+1}(1) \circ t_n^i(1) \sim t_n^{i+1}(1) \circ t_n^i(1) \circ t_n^{i+1}(1)$ ;
- (10) cup – cup isotopy :  $g_{n+2}^{i+k} \circ g_n^i \sim g_{n+2}^i \circ g_n^{i+k-2}$ ;
- (11) cap – cap isotopy :  $f_n^{i+k-2} \circ f_{n+2}^i \sim f_n^i \circ f_{n+2}^{i+k}$ ;
- (12) cup – cap isotopy :  $g_n^{i+k-2} \circ f_n^i \sim f_{n+2}^i \circ g_{n+2}^{i+k}$ ,  $g_n^i \circ f_n^{i+k-2} \sim f_{n+2}^{i+k} \circ g_{n+2}^i$ ;
- (13) cup – crossing isotopy :  $g_n^i \circ t_{n-2}^{i+k-2}(l) \sim t_n^{i+k}(l) \circ g_n^i$ ,  
 $g_n^{i+k} \circ t_{n-2}^i(l) \sim t_n^i(l) \circ g_n^{i+k}$ ;
- (14) cap – crossing isotopy :  $f_n^i \circ t_{n-2}^{i+k}(l) \sim t_n^{i+k-2}(l) \circ f_n^i$ ,  
 $f_n^{i+k} \circ t_{n-2}^i(l) \sim t_{n-2}^i(l) \circ f_n^{i+k}$ ;
- (15) crossing – crossing isotopy :  $t_n^i(l) \circ t_n^{i+k}(m) \sim t_n^{i+k}(m) \circ t_n^i(l)$ ;
- (16) pitchfork move :  $t_n^i(1) \circ g_n^{i+1} \sim t_n^{i+1}(2) \circ g_n^i$ ,  $t_n^i(2) \circ g_n^{i+1} \sim t_n^{i+1}(1) \circ g_n^i$ ;

where  $k \geq 2$  and  $1 \leq l, m \leq 2$ .

We can reformulate Lemma 2 in the following way:

**Lemma-Definition 2.** *Define the category  $CTan$  of combinatorial tangles where objects are natural numbers, and morphisms are classes of combinatorial diagrams modulo the relations of Lemma 2. Then the categories  $Tan_{iso}$  and  $CTan$  are equivalent.*

In fact, these two categories are not only equivalent, but truly isomorphic.

**3.2. Affine tangles.** Define a smooth affine  $(n, m)$  tangle as a proper, smooth embedding of  $(n+m)/2$  arcs and a finite number of circles into  $\mathbb{R} \times S$ , where  $S \subset \mathbb{C}$  is the annulus  $\{z \in \mathbb{C} | 1 \leq |z| \leq 2\}$ , such that the boundary points of the arcs map bijectively on the  $n+m$  points  $((0, \zeta_{n+1}), (0, \zeta_{n+1}^2), \dots, (0, \zeta_{n+1}^n), (0, 2\zeta_{m+1}), (0, 2\zeta_{m+1}^2), \dots, (0, 2\zeta_{m+1}^m))$ , where  $\zeta_k^k = 1$ , and the arcs are orthogonal to the boundary at the ends. From now on we will call the tangles on a segment *linear tangles* to distinguish them from the affine case. Note that any point  $\theta$  on

the unit circle which is not a root of unity defines a functor from the category of linear tangles to the category of affine tangles induced by a map  $\mathbb{R} \simeq S - \theta$ .

The description of generators and relations can be easily extended to affine tangles. We add two generators  $r_n$  and  $r'_n$  which correspond to the counterclockwise and clockwise shifts of all strands respectively. Then elements  $g_n^n$ ,  $f_n^n$  and  $t_n^n(l)$  may be defined by

$$(17) \quad g_n^n = r'_n \circ g_n^{n-1} \circ r_{n-2};$$

$$(18) \quad f_n^n = r'_{n-2} \circ f_n^{n-1} \circ r_n;$$

$$(19) \quad t_n^n(l) = r'_n \circ t_n^{n-1}(l) \circ r_n.$$

They depend on other generators and are used only to show that the set of generators already described is sufficient.

**Lemma 3.** *Elements  $g_n^i$ ,  $f_n^i$ ,  $t_n^i(l)$ , where  $i = 1, \dots, n$ ,  $r_n$  and  $r'_n$  generate the category of affine tangles up to isotopies.*

*Proof.* Divide an affine tangle diagram by concentric circles into thin annuli so that each annulus contains only one critical or crossing point. Enumerate the points of intersection of strands and circles in an arbitrary way and adjust the strands near the circles so that they are orthogonal. Then each annulus is a diagram of a composition of a  $g_n^i$ ,  $f_n^i$  or  $t_n^i(l)$  with some power of  $r_{n\pm 2}$  or  $r'_{n\pm 2}$ .  
□.

To proceed, we need to refine our approach to tangle isotopies.

**Definition 1.** *Define a 2-category **Tan** (resp. **CTan**, resp. **ATan**) as having natural numbers as objects,  $(n, m)$  tangles (resp. combinatorial tangles, resp. affine tangles) as 1-morphisms, and classes of cobordisms of tangles (resp. combinatorial tangles, resp. affine tangles) up to isotopy, as 2-morphisms.*

Let us call two isotopies of tangles equivalent, if they are isotopic as tangle cobordisms.

**Definition 2.** *Let us call an isotopy of affine tangles  $\alpha_1 \sim \alpha_2$  linear, if it is a closure of an image of an isotopy of linear tangles  $\beta_1 \sim \beta_2$  under a map  $\phi \times id \times id : (\mathbb{R} \times (0, 1)) \times [0, 1] \times [0, 1] \rightarrow S \times [0, 1] \times [0, 1]$ , where  $\phi$  is a diffeomorphism of  $\mathbb{R} \times (0, 1)$  onto a dense subset of  $S$ .*

Here closure means that  $\phi$  may send some ends of arcs to inner points of  $S$ , and in this situation we require that the closure of  $\phi(\beta_i)$  set-theoretically coincides with  $\alpha_i$ .

**Lemma 4.** *Any isotopy of affine tangles is equivalent to a composition of two linear isotopies.*

*Proof.* It follows from the topological fact that the inside of  $S \times [0, 1] \times [0, 1]$  is homeomorphic to the inside of  $S \times [0, 1] \times [0, 1] - ([1, 2] \times [0, 1] \times [0, 2/3] \sqcup [\zeta_{n+1}^{3/2}, 2\zeta_{n+1}^{3/2}] \times [0, 1] \times [1/3, 1])$ .

The points of intersection of arcs of the initial tangle with  $[1, 2] \times [0, 1]$  (resp. the final tangle with  $[\zeta_{n+1}^{3/2}, 2\zeta_{n+1}^{3/2}] \times [0, 1]$ ) are retracted on the inserted pieces of the boundary and may be viewed as new fixed endpoints for the tangles which hereby become linear.

□

An extended list contains, first, the relations (6)-(16), where the indices run through  $1, \dots, n$  modulo  $n$ , and, second, the commutation relations for  $r_n, r'_n$  with all other generators. In fact, it suffices to add the latter.

**Claim 4.** *Any isotopy of affine tangles is equivalent to a composition of a finite number of elementary isotopies from Lemma 2 and the following elementary affine isotopies.*

$$(20) \quad r_n \circ r'_n \sim id \sim r'_n \circ r_n$$

$$(21) \quad r'_{n-2} \circ f_n^i \circ r_n \sim f_n^{i+1}, \quad i = 1, \dots, n-2; \quad f_n^{n-1} \circ (r_n)^2 \sim f_n^1;$$

$$(22) \quad r'_n \circ g_n^i \circ r_{n-2} \sim g_n^{i+1}, \quad i = 1, \dots, n-2; \quad (r'_n)^2 \circ g_n^{n-1} \sim g_n^1;$$

$$(23) \quad r'_n \circ t_n^i(l) \circ r_n \sim t_n^{i+1}(l); \quad (r'_n)^2 \circ t_n^{n-1}(l) \circ (r_n)^2 \sim t_n^1(l).$$

*Proof.* By Lemma 4, any isotopy of affine tangles is a composition of two linear isotopies, which in turn are, by Lemma 2, compositions of elementary isotopies (6) - (16), where indices run through  $1, \dots, n$  modulo  $n$ . Then it suffices to prove that the relations (6) - (16) with one of the indices equal to  $n$  are derived from relations (6) - (16) with indices  $1, \dots, n-1$ , together with relations (20) - (23), which may be done by direct computation.

□

This Claim may be rewritten in the language of combinatorial tangles in the following way:

**Lemma-Definition 3.** *Define the category  $ACTan$  of affine combinatorial tangles where objects are natural numbers, and morphisms are classes of combinatorial diagrams of affine tangles modulo the relations of Claim 4. Then the categories  $ATan_{iso}$  and  $ACTan$  are equivalent.*

*Proof.* There is a functor  $ACDiag \rightarrow ATan_{iso}$ . By Lemma 3 this functor is surjective on morphisms. Since elementary isotopies are isotopies, this functor factors through  $ACTan$ . Claim 4 implies that if two affine tangles  $\alpha$  and  $\beta$  are isotopic, then there exists another isotopy  $\alpha \sim \beta$ , which is a composition of elementary isotopies. Thus, the functor  $ACTan \rightarrow ATan_{iso}$  is injective on morphisms.

□

Once again, we get not only an equivalence, but an isomorphism of categories.

**3.3. Framed tangles.** All preceding constructions may be carried out for framed tangles. Define the generators  $\hat{g}_n^i$  (resp.  $\hat{f}_n^i$ , resp.  $\hat{t}_n^i(l)$ , resp.  $\hat{r}_n^i$ ) as tangles  $g_n^i$  (resp.  $f_n^i$ , resp.  $t_n^i(l)$ , resp.  $r_n^i$ ) with blackboard framing. Introduce new generators  $\hat{w}_n^i(1)$  and  $\hat{w}_n^i(2)$ , which correspond to positive and negative twists of framing of the  $i$ th strand of an  $(n, n)$  identity tangle. Define the category  $AFDiag$  of affine framed tangle diagrams, and consider a functor from  $AFDiag$  to the category  $AFTan_{iso}$  of affine framed tangles up to isotopy.

The relations for framed tangles are transformed as follows: first, relations (6), (8)-(16), (20)-(23) remain unchanged:



- $$\begin{aligned}
 (24) \quad & \hat{f}_n^i \circ \hat{g}_n^{i+1} \sim id \sim \hat{f}_n^{i+1} \circ \hat{g}_n^i; \\
 (25) \quad & \hat{t}_n^i(2) \circ \hat{t}_n^i(1) \sim id \sim \hat{t}_n^i(1) \circ \hat{t}_n^i(2); \\
 (26) \quad & \hat{t}_n^i(2) \circ \hat{t}_n^{i+1}(2) \circ \hat{t}_n^i(2) \sim id \sim \hat{t}_n^i(1) \circ \hat{t}_n^{i+1}(1) \circ \hat{t}_n^i(1); \\
 (27) \quad & \hat{g}_{n+2}^{i+k} \circ \hat{g}_n^i \sim \hat{g}_{n+2}^i \circ \hat{g}_n^{i+k-2}; \\
 (28) \quad & \hat{f}_n^{i+k-2} \circ \hat{f}_{n+2}^i \sim \hat{f}_n^i \circ \hat{f}_{n+2}^{i+k}; \\
 (29) \quad & \hat{g}_n^{i+k-2} \circ \hat{f}_n^i \sim \hat{f}_{n+2}^i \circ \hat{g}_{n+2}^{i+k}, \quad \hat{g}_n^i \circ \hat{f}_n^{i+k-2} \sim \hat{f}_{n+2}^{i+k} \circ \hat{g}_{n+2}^i; \\
 (30) \quad & \hat{g}_n^i \circ \hat{t}_{n-2}^{i+k-2}(l) \sim \hat{t}_n^{i+k}(l) \circ \hat{g}_n^i, \quad \hat{g}_n^{i+k} \circ \hat{t}_{n-2}^i(l) \sim \hat{t}_n^i(l) \circ \hat{g}_n^{i+k}; \\
 (31) \quad & \hat{f}_n^i \circ \hat{t}_n^{i+k}(l) \sim \hat{t}_{n-2}^{i+k-2}(l) \circ \hat{f}_n^i, \quad \hat{f}_n^{i+k} \circ \hat{t}_n^i(l) \sim \hat{t}_{n-2}^i(l) \circ \hat{f}_n^{i+k}; \\
 (32) \quad & \hat{t}_n^i(l) \circ \hat{t}_n^{i+k}(m) \sim \hat{t}_n^{i+k}(m) \circ \hat{t}_n^i(l); \\
 (33) \quad & \hat{t}_n^i(1) \circ \hat{g}_n^{i+1} \sim \hat{t}_n^{i+1}(2) \circ \hat{g}_n^i, \quad \hat{t}_n^i(2) \circ \hat{g}_n^{i+1} \sim \hat{t}_n^{i+1}(1) \circ \hat{g}_n^i; \\
 (34) \quad & \hat{r}_n \circ \hat{r}'_n \sim id \sim \hat{r}'_n \circ \hat{r}_n \\
 (35) \quad & \hat{r}'_{n-2} \circ \hat{f}_n^i \circ \hat{r}_n \sim \hat{f}_n^{i+1}, \quad i = 1, \dots, n-2; \quad \hat{f}_n^{n-1} \circ (\hat{r}_n)^2 \sim \hat{f}_n^1; \\
 (36) \quad & \hat{r}'_n \circ \hat{g}_n^i \circ \hat{r}_{n-2} \sim \hat{g}_n^{i+1}, \quad i = 1, \dots, n-2; \quad (\hat{r}'_n)^2 \circ \hat{g}_n^{n-1} \sim \hat{g}_n^1; \\
 (37) \quad & \hat{r}'_n \circ \hat{t}_n^i(l) \circ \hat{r}_n \sim \hat{t}_n^{i+1}(l); \quad (\hat{r}'_n)^2 \circ \hat{t}_n^{n-1}(l) \circ (\hat{r}_n)^2 \sim \hat{t}_n^1(l).
 \end{aligned}$$

The relation (7) turns into

$$(38) \quad \hat{f}_n^i \circ \hat{t}_n^{i\pm 1}(l) \circ \hat{g}_n^i \sim \hat{w}_n^i(l).$$

Then new relations for twists are added:

- $$\begin{aligned}
 (39) \quad & \hat{w}_n^i(1) \circ \hat{w}_n^i(2) \sim id, \quad \hat{w}_n^i(l) \circ \hat{w}_n^j(k) \sim \hat{w}_n^j(k) \circ \hat{w}_n^i(l), \quad i \neq j; \\
 (40) \quad & \hat{w}_n^i(k) \circ \hat{g}_n^i \sim \hat{w}_n^{i+1}(k) \hat{g}_n^i, \quad \hat{w}_n^i(k) \circ \hat{g}_n^j \sim \hat{g}_n^j \circ \hat{w}_n^{i+1\pm 1}(k), \quad i \neq j, j+1; \\
 (41) \quad & \hat{f}_n^i \circ \hat{w}_n^i(k) \sim \hat{f}_n^i \circ \hat{w}_n^{i+1}(k), \quad \hat{w}_n^i(k) \circ \hat{f}_n^j \sim \hat{f}_n^j \circ \hat{w}_n^{i-1\pm 1}(k), \quad i \neq j, j+1 \\
 (42) \quad & \hat{w}_n^i(k) \circ \hat{t}_n^i \sim \hat{w}_n^{i+1}(k) \hat{t}_n^i, \quad \hat{w}_n^i(k) \circ \hat{t}_n^j \sim \hat{t}_n^j \circ \hat{w}_n^i(k), \quad i \neq j, j+1; \\
 (43) \quad & \hat{t}_n^i \circ \hat{w}_n^i(k) \sim \hat{f}_n^i \circ \hat{w}_n^{i+1}(k), \quad \hat{w}_n^i(k) \circ \hat{f}_n^j \sim \hat{t}_n^j \circ \hat{w}_n^i(k), \quad i \neq j, j+1 \\
 (44) \quad & \hat{w}_n^i(k) \circ \hat{r}_n \sim \hat{r}_n \circ \hat{w}_n^{i-1}(k), \quad \hat{w}_n^i(k) \circ \hat{r}'_n \sim \hat{r}'_n \circ \hat{w}_n^{i+1}(k);
 \end{aligned}$$

**Claim 5.** *Any isotopy of affine framed tangles is equivalent to a composition of elementary isotopies (24)-(44).*

*Proof.* There is a forgetful functor from the 2-category of framed functors and their isotopies to the 2-category of non-framed tangles and their isotopies, which forgets the framing. Thus, for every isotopy there is a composition of (24)-(38), which differs only in framing, and that can be ruled out by the commutation laws (39)-(44) of twists with all other generators.

□.

This Claim may be rewritten in the language of combinatorial tangles in the following way:

**Lemma-Definition 4.** *Define the category AFCTan of affine framed combinatorial tangles where objects are natural numbers, and morphisms are classes of*

*combinatorial diagrams of affine framed tangles modulo the relations of Claim 5. Then the categories  $AFTan_{iso}$  and  $AFCTan$  are equivalent.*

Once again, there exists not only an equivalence, but a genuine isomorphism of categories.

#### 4. FUNCTORS ASSIGNED TO TANGLES

For Springer fibers over  $Z_{2n}$  the braid group actions for different  $n$  can be linked together to construct a weak representation of the category of (even-stranded) framed affine tangles. This happens because in this particular case the varieties  $P_{k,Z_{2n}}$  of partial flags compatible with the action of  $Z_{2n}$  are isomorphic to  $B_{Z_{2n-2}}$  for all  $k$  (see Lemma 5 below). Denote  $B_{2n} = B_{Z_{2n}}$ ,  $U_{2n} = U_{Z_{2n}}$ ,  $\mathcal{D}_{2n} = \mathcal{D}_{Z_{2n}}$ . Then the spherical functors from section 2.3 act between categories  $\mathcal{D}_{2n-2}$  and  $\mathcal{D}_{2n}$ .

In [4] Cautis and Kamnitzer build a weak categorification of tangle calculus using derived categories of coherent sheaves on certain compactifications of  $U_{2n}$ . So far our calculus is a restriction of theirs, but we argue that the functors that constitute this weak representation are best described as a representation of a category of framed tangles, not oriented tangles. A twist of the framing corresponds to a shift in the triangulated category. The details are carried out in the rest of the section below.

**Lemma 5.**  $P_{k,Z_{2n}} \simeq B_{Z_{2n-2}}$ .

*Proof.* By definition

$$P_{k,Z_{2n}} \cong \{(0 = V_0 \subset V_1 \subset \dots \hat{V}_k \subset \dots \subset V_n = V) | \\ Z_{2n}V_{k+1} \subset V_{k-1}, Z_{2n}V_i \subset V_{i-1}, i \neq k+1\}.$$

Fix a flag  $(0 = V_0 \subset V_1 \subset \dots \hat{V}_k \subset \dots \subset V_n = V) \in P_{k,Z_{2n}}$ . As  $\dim \text{Ker } Z_{2n} = 2$ , we necessarily have  $Z_{2n}V_{k+1} = V_{k-1}$ , hence  $\text{Ker } Z_{2n} \subset V_{k+1}$  and  $Z_{2n}$  drops the dimension of  $V_{k+2}, \dots, V_{2n}$  by 2 exactly. Then we can map this flag to a flag  $(V_0 \subset \dots \subset V_{k-1} = Z_{2n}V_{k+1} \subset Z_{2n}V_{k+2} \subset \dots \subset Z_{2n}V_{2n}) \in B_{Z_{2n-2}}$ . The inverse map is obvious.

□

It follows that  $U_{2n-2}$  is isomorphic to  $T^*P_k \times_{\mathfrak{g}} S_{Z_{2n}}$ , because the map  $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$  is finite outside  $\mathcal{N}$ . This isomorphism induces an isomorphism of derived categories of coherent sheaves, and the functor  $\Phi_{2n}^k$  (see the end of the previous section) by an abuse of notation will be considered to have  $\mathcal{D}_{2n-2}$  as its initial category.

Define a map  $\Phi$  which assigns a functor  $\mathcal{D}_{2m} \rightarrow \mathcal{D}_{2n}$  to each combinatorial framed  $(2m, 2n)$  tangle in the following way:

- (1) assign the functor  $G_{2n}^k := \Phi_k^{2n}$  to  $\hat{g}_{2n}^k$ ;
- (2) assign the functor  $F_{2n}^k := (\Phi_k^{2n})^R[1] \cong (\Phi_k^{2n})^L[-1]$  to  $\hat{f}_{2n}^k$ ;
- (3) assign the functor  $T_{2n}^k(1) := Tw_{2n}^k|_{U_{2n}}$  to the positive non-twisted crossing  $\hat{t}_{2n}^k(1)$  and the functor  $T_{2n}^k(2) := Tw_{2n}^k'|_{U_{2n}}$  to the negative non-twisted crossing  $\hat{t}_{2n}^k(2)$ ;
- (4) assign the shift  $[1]$  to a positive twist  $\hat{w}_{2n}^k(1)$  of any strand and the shift  $[-1]$  to the negative twist  $\hat{w}_{2n}^k(2)$ .

Note that this functors  $\Phi$  differs slightly from the functors  $\Psi$  defined in [4]. The conventional difference is that we read tangle diagrams from bottom to top, and switch 1 and 2 in the notation for  $t_n^i(1)$ ,  $t_n^i(2)$ . The essential difference is that

Cautis and Kamnitzer consider oriented tangles, hence there are two more possible crossings  $t_n^i(3)$  and  $t_n^i(4)$  obtained by switching the orientation of one strand in  $t_n^i(1)$ ,  $t_n^i(2)$  respectively. Our correspondence  $\Phi$  differs from their correspondence  $\Psi$  by the shift by  $[\pm 1]$ :

$$\begin{aligned}\Psi(t_n^i(1)) &= \Phi(t_n^i(2))[1]; & \Psi(t_n^i(2)) &= \Phi(t_n^i(1))[-1]; \\ \Psi(t_n^i(3)) &= \Phi(t_n^i(2))[-1]; & \Psi(t_n^i(4)) &= \Phi(t_n^i(1))[1].\end{aligned}$$

Switching from  $\Psi$  to  $\Phi$  does not affect the relations (6), (8)-(15), the relation (16) gets identical shifts on both sides, and the relation (7) turns into (38), which suits us very well.

**Proposition 4.** *The framed tangle relations (24)-(33), (38)-(44) hold for functors  $\Phi(*)$ , where  $*$  =  $\hat{g}_{2n}^k, \hat{f}_{2n}^k, \hat{t}_{2n}^k(l), \hat{w}_{2n}^k$ .*

*Proof.* This result for ordinary tangles immediately follows from [4]. The relations for twists are trivial, since all functors commute with shifts.

□

It remains to define functors for  $r_{2n}$ ,  $r'_{2n}$  and prove relations (34)-(37). This will be done in the next subsection.

**4.1. Affine generators.** In order to end up with a nice description of morphism spaces, we should twist some of the functors by a trivial, but not canonically trivial, line bundle  $\mathcal{L} := \mathcal{O} \otimes_{\mathbb{C}} L$ , where  $L$  is a 1-dimensional vector space with a property that  $L^{\otimes 2} \cong \Lambda^2 \text{Ker } Z_{2n}$ .

Define a functor  $\mathcal{S}_{2n} = \mathcal{E}_{2n}^{\vee} \otimes \mathcal{L} \otimes \cdot$ , which will be proven to correspond to a frame of an affine braid which leaves strands  $1, \dots, 2n-1$  in place and carries strand  $2n$  around the circle counterclockwise, passing beneath other strands.

**Lemma 6.** *The following relations hold:*

- (1)  $\mathcal{S}_{2n} \circ G_{2n}^i \cong G_{2n}^i \circ \mathcal{S}_{2n}, \quad \mathcal{S}_{2n} \circ F_{2n}^i \cong F_{2n}^i \circ \mathcal{S}_{2n},$   
 $\mathcal{S}_{2n} \circ T_{2n}^i(l) \cong T_{2n}^i(l) \circ \mathcal{S}_{2n}, \quad i = 1, \dots, 2n-2;$
- (2)  $F_{2n}^{2n-1} \circ \mathcal{S}_{2n} \circ T_{2n}^{2n-1}(2) \circ \mathcal{S}_{2n} \circ T_{2n}^{2n-1}(2) \cong F_{2n}^{2n-1};$
- (3)  $\mathcal{S}_{2n} \circ T_{2n}^{2n-1}(2) \circ \mathcal{S}_{2n} \circ T_{2n}^{2n-1}(2) \circ G_{2n}^{2n-1} \cong G_{2n}^{2n-1};$

*Proof.* The first part is obvious since tensoring with  $\mathcal{E}_{2n}^{\vee}$  commutes with all maps from definitions of  $G_{2n}^i$  for  $i = 1, \dots, 2n-2$ . The rest is proved by a direct computation. Let us omit the indices for shortness: denote  $G_{2n}^{2n-1}$  by  $G$ ,  $F_{2n}^{2n-1}$  by  $F$ ,  $T_{2n}^{2n-1}(1)$  by  $T$ ,  $T_{2n}^{2n-1}(2)$  by  $T'$ ,  $\mathcal{S}_{2n}$  by  $\mathcal{S}$ . Denote by  $D_{2n-1}$  the divisor (5) for  $k = 2n-1$ . Then  $\mathcal{E}_{2n} \otimes \mathcal{E}_{2n-1}|_{D_{2n-1}} \cong \Lambda^2(\mathcal{V}/\mathcal{V}_{2n-2}|_{D_{2n-1}}) \cong \mathcal{O}_{D_{2n-1}} \otimes \Lambda^2 \text{Ker } Z_{2n}$ . Recall that  $F \cong G^L[-1] \cong G^R[1]$  and  $T' \cong \{id \rightarrow G \circ G^L\} \cong \{id \rightarrow G \circ F[1]\}$ . Recall also that with  $i : D_{2n-1} \rightarrow U_{2n}$  and  $\pi : D_{2n-1} \rightarrow U_{2n-2}$  we have  $G \cong i_*(\mathcal{E}_{2n-1} \otimes \pi^*)$ . The forgetful map  $p : D_{2n-1} \rightarrow \mathbb{P}(\mathcal{V}_{2n-1}/Z_{2n}V) \simeq \mathbb{P}^1$  decomposes  $D_{2n-1}$  into a direct product  $D_{2n-1} \simeq \mathbb{P}^1 \times U_{2n-2}$ , and on  $D_{2n-1}$  we have  $\mathcal{E}_{2n}^{\vee} \cong p^*T_{\mathbb{P}^1}^{\vee}(1)$  and  $\mathcal{E}_{2n-1} \cong p^*\mathcal{O}(-1)$ . Moreover,  $\mathcal{O}_{D_{2n-1}}(D_{2n-1}) \cong p^*T_{\mathbb{P}^1}^{\vee}$ . Then

$$\begin{aligned}(45) \quad F \circ \mathcal{S} \circ T' \circ \mathcal{S} &\cong F(\mathcal{E}_{2n}^{\vee} \otimes \mathcal{L} \otimes T'(\cdot \otimes \mathcal{E}_{2n}^{\vee} \otimes \mathcal{L})) \cong \\ &\cong \{\pi_*((\mathcal{E}_{2n}^{\vee})^{\otimes 3} \otimes \mathcal{L}^{\otimes 2} \otimes i^*) \rightarrow \pi_*((\mathcal{E}_{2n}^{\vee})^{\otimes 2} \otimes \mathcal{L} \otimes i^* i_*(\mathcal{E}_{2n-1} \otimes \pi^* \pi_*((\mathcal{E}_{2n}^{\vee})^{\otimes 2} \otimes \mathcal{L} \otimes i^*))) [1]] [-1]\end{aligned}$$

Furthermore,  $\mathcal{O}(-D_{2n-1})[-1] \rightarrow i^*i_* \rightarrow id$ , and  $\pi_*(\mathcal{E}_{2n}^\vee \otimes \pi^*) \simeq 0$ , hence the above expression is isomorphic to

$$\begin{aligned}
 (46) \quad & \{\pi_*((\mathcal{E}_{2n}^\vee)^{\otimes 3} \otimes \mathcal{L}^{\otimes 2} \otimes i^*) \rightarrow \pi_*((\mathcal{E}_{2n}^\vee) \otimes 2 \otimes \mathcal{E}_{2n-1} \otimes \mathcal{L} \otimes \pi^* \pi_*((\mathcal{E}_{2n}^\vee)^{\otimes 2} \otimes \mathcal{L} \otimes i^*)) [1]\} [-1] \cong \\
 & \{\pi_*((\mathcal{E}_{2n}^\vee)^{\otimes 3} \otimes \mathcal{L}^{\otimes 2} \otimes i^*) \rightarrow \pi_*((\mathcal{E}_{2n}^\vee)^{\otimes 2} \otimes \mathcal{L}^{\otimes 2} \otimes i^*) \otimes (\text{Ker } Z_{2n})^\vee\} [-1] \cong \\
 & \cong \pi_* \mathcal{E}_{2n-1}^\vee \otimes (\mathcal{E}_{2n}^\vee)^{\otimes 2} \otimes \mathcal{L}^{\otimes 2} \otimes i^* [-1] \cong \pi_* \mathcal{E}_{2n}^\vee \otimes i^* [-1] \cong F[-1] \cong FT.
 \end{aligned}$$

This proves (2). The remaining relation (3) is proved the same way.

□

Assign a functor to the affine generator  $\hat{r}_{2n}$  by

$$(47) \quad R_{2n} := \Phi(\hat{r}_{2n}) := S_{2n} \circ T_{2n}^{2n-1}(2) \circ \dots \circ T_{2n}^1(2).$$

This functor is invertible, hence we can assign its inverse  $R'_{2n}$  to the generator  $r'_{2n}$ .

**Proposition 5.** *Relations (34)-(37) hold for functors  $G_{2n}^i$ ,  $F_{2n}^i$ ,  $T_{2n}^i(l)$ ,  $R_{2n}$ ,  $R'_{2n}$ .*

*Proof.* Substitute the definition of  $R_{2n}$ ,  $R'_{2n}$  into (34)-(37). Then using commutation relations and relations from Lemma 6 the functors  $\mathcal{S}$  may be excluded from the expressions, thus reducing the statement to the non-affine case.

□

We have constructed a weak representation of the category  $AFCTan$  of affine framed combinatorial tangles. Using the equivalence of categories  $AFCTan \sim AFTan_{iso}$  from Lemma-Definition 4, we can view it as a weak representation of the category  $AFTan_{iso}$  of affine framed tangles up to isotopy.

## 5. THE EXOTIC $t$ -STRUCTURE

According to Bezrukavnikov and Mirkovic ([2], [3]), the category  $\mathcal{D}_{2n}$  has a distinguished  $t$ -structure, called the *exotic*  $t$ -structure, which may be described as follows.

There is an action of the affine braid group  $ABr_{2n}$  on  $D^b(\tilde{\mathcal{N}})$ , which may be transferred to  $\mathcal{D}_{2n}$ . Then the  $t$ -structure is characterized by

$$(48) \quad \{\alpha \in D^{\geq 0}\} \iff \{\forall b^+ \in ABr_{2n}^+ \ R\Gamma(\Psi(b^+)\alpha) \in D^{\geq 0}(Vect)\};$$

$$(49) \quad \{\alpha \in D^{\leq 0}\} \iff \{\forall b^- \in ABr_{2n}^+ \ R\Gamma(\Psi(b^-)\alpha) \in D^{\leq 0}(Vect)\}.$$

**Claim 6.** *The functors  $G_{2n}^i$  are exact with respect to this  $t$ -structure.*

*Proof.* Since all  $G_{2n}^i$  are conjugate via  $t$ -exact invertible functors  $R_{2n}$ , it is enough to prove that one of them is exact. In fact, we will prove that  $G_{2n}^{2n-1}$  is right exact and  $G_{2n}^1$  is left exact. To prove this, we will show that  $(G_{2n}^1)^L$  is right exact and  $(G_{2n}^{2n-1})^R$  is left exact.

Let  $G = G_{2n}^k$  for some  $k$ . By definition of the  $t$ -structure we know that  $T = \{GG^R \rightarrow id\}$  is left exact and  $T'$  right exact. Then from triangles  $GG^R \rightarrow id \rightarrow T$  and  $T' \rightarrow id \rightarrow GG^L$  we conclude that  $GG^R$  is left exact and  $GG^L$  right exact.

Furthermore, for any  $\alpha \in \mathcal{D}^0$  and any  $\beta$  such that  $G^L\beta = \mathcal{O}_{U_{2n-2}}$  we have

$$(50) \quad R\Gamma(G^R\alpha) \cong RHom^i(G^L\beta, G^R\alpha) \cong RHom^i(\beta, GG^R\alpha).$$

Let  $k = 2n - 1$ .

Consider  $\beta = \mathcal{E}_{2n}[-1]$ . Then

$$(51) \quad R\Gamma^i(G^R\alpha) \cong R\Gamma^i(\mathcal{E}_{2n}^\vee[1] \otimes GG^R\alpha).$$

The functor  $\mathcal{E}_{2n}^\vee \otimes \cdot$  represents an affine braid which leaves strands number  $1, \dots, 2n-1$  in place and leads strand  $2n$  around the circle counterclockwise, passing underneath. It is a positive braid, and  $\mathcal{E}_{2n}^\vee \otimes T'_{2n-1}$  is a positive braid. Then from the triangle

$$(52) \quad \mathcal{E}_{2n}^\vee \otimes T'\alpha \rightarrow \mathcal{E}_{2n}^\vee \otimes \alpha \rightarrow \mathcal{E}_{2n}^\vee \otimes GG^R[2]\alpha$$

we conclude that  $\mathcal{E}_{2n}^\vee \otimes GG^R[2]\alpha \in \mathcal{D}^{\geq -1}$ , hence  $\mathcal{E}_{2n}^\vee \otimes GG^R[1]\alpha \in \mathcal{D}^{\geq 0}$ , q.e.d.

Now let  $k = 1$ . Choose  $\beta = \mathcal{E}_1$ . Then

$$(53) \quad R\Gamma^i(G^L\alpha) \cong R\Gamma^i(\mathcal{E}_1^\vee \otimes GG^L\alpha).$$

The functor  $\mathcal{E}_1^\vee \otimes \cdot$  corresponds to the braid that leaves strands  $2, \dots, 2n$  in place and leads strand 1 around the circle counterclockwise, passing above other strands. It is a negative braid, so we see that  $\mathcal{E}_1^\vee \otimes GG^L\alpha \in \mathcal{D}^{\leq 0}$  right away, q.e.d.

□

## 6. IRREDUCIBLE OBJECTS.

Consider now affine  $(0, 2n)$  tangles. A combinatorial affine framed tangle  $\alpha$  corresponds to a functor  $\Phi(\alpha) : \mathcal{D}_0 \rightarrow \mathcal{D}_{2n}$ , and  $\mathcal{D}_0 \simeq D^b(Vect)$ . Denote by  $E_\alpha$  the image in  $\mathcal{D}_{2n}$  of the 1-dimensional vector space. For isotopic tangles  $\alpha_1$  and  $\alpha_2$  the objects  $E_{\alpha_1}, E_{\alpha_2}$  are isomorphic. In general, different isotopies give rise to isomorphisms that differ by a scalar multiplication, but it is possible to distinguish a system of "nice" isotopies in such a way that for nicely isotopic tangles  $\alpha_1$  and  $\alpha_2$  the objects  $E_{\alpha_1}$  and  $E_{\alpha_2}$  would be canonically isomorphic. If  $\alpha$  is a crossingless matching, then  $\Phi(\alpha)$  is a composition of generators  $g_{2m}^k$  for various  $m$  and  $k$ , and by Claim 6  $\Phi(\alpha)$  is exact, hence  $E_\alpha$  lies in the heart of the exotic  $t$ -structure on  $\mathcal{D}_{2n}$ . It will turn out (cf. Theorem 2) that these objects are precisely the irreducible objects of  $\mathcal{A} - mod$ .

For a  $(0, 2n)$  tangle  $\alpha$  denote by  $\alpha^\vee$  its image under inversion (for linear tangles, the analogous operation would be the mirror reflection with respect to the horizontal axis). It is a  $(2n, 0)$  tangle.

**Lemma 7.**  $\Phi(\alpha)^R \cong \Phi(\alpha^\vee)[-n]$ .

*Proof.* This follows from  $\Phi(g_{2n}^k)^R \cong \Phi(f_{2n}^k)[-1]$  and  $\Phi(t_{2n}^k(1))^R \cong \Phi(t_{2n}^k(2))$ .

□

**Lemma 8.** *There are  $\binom{2n}{n}$  affine crossingless matchings of  $2n$  points. They are indexed by arrangements of  $n$  pluses and  $n$  minuses on  $2n$  places around a circle.*

*Proof.* Having a circle of pluses and minuses, connect each plus to the nearest minus (counting clockwise) with the property that there is the same number of pluses and minuses between them. Such a minus must exist, because total numbers of pluses and minuses are equal. To construct an inverse map, notice that for every arc in a crossingless matching we can define a "clockwise" orientation, which is isotopy invariant. Then we can mark all sources by pluses and all targets by minuses.

□

**Remark.** ([13]) Crossingless matchings of  $2n$  points also naturally label indecomposable self-dual projective modules in the principal block of parabolic category  $\mathcal{O}$  for the Lie algebra  $\mathfrak{gl}_n$  with respect to the parabolic with Levi subalgebra  $\mathfrak{gl}_n \times \mathfrak{gl}_n$  ([11, Theorem 5.2.4]). The endomorphism ring of the direct sum of all these indecomposable modules (as well as its quasi-hereditary cover) has a graphical description similar to ours ([11, Theorem 5.3.1], [11, Proposition 5.6.2]). Moreover, the corresponding Springer fiber appears naturally, namely its cohomology is isomorphic to the center of either of these endomorphism rings ([11, Theorem 1] and [?]). Different parabolic category  $\mathcal{O}$ 's were used in [12] to give a representation theoretic version of [4].

A link  $\gamma$  corresponds to a functor  $\Phi(\gamma) : \mathcal{D}_0 \rightarrow \mathcal{D}_0$ , which necessarily is a tensor multiplication by a complex  $E_\gamma$  of vector spaces. A flat link is a union of separate loops, each of them may or may not enclose the origin. Call the loop a 1-loop if it encloses the origin, and a 0-loop if it does not. For a link  $\gamma$  denote by  $n_0(\gamma)$  the number of 0-loops in  $\gamma$  and by  $n_1(\gamma)$  the number of 1-loops. Since the equivalence class of functors is isotopy invariant, the cohomology of  $E_\gamma$  is also isotopy invariant, and moreover is a tensor product of cohomological spaces corresponding to single loops (because if loops are not entangled, they may be separated by a circle, presenting the corresponding functor as a composition). From Example 7.1 we see that the space corresponding to an 0-loop  $\gamma_0$  has two 1-dimensional components in degrees 0 and 2, and the space corresponding to a 1-loop  $\gamma_1$  has one 2-dimensional component in degree 1. Let us denote their shifts  $H^*(E_{\gamma_0[1]}) = \mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_1$  and  $H^*(E_{\gamma_1[1]}) = \mathcal{A}_0$  respectively, as in the Example cited above..

By Lemma 7 for two tangles  $\alpha, \beta$  holds

$$(54) \quad \text{Hom}^*(E_\alpha, E_\beta) \simeq (\Phi(\alpha))^R E_\beta \cong H^*(\Phi(\alpha^\vee) E_\beta[-n]) \cong H^*(E_{\alpha^\vee \circ \beta}[-n]).$$

If  $\alpha_1$  and  $\alpha_2$  are two flat  $(0, 2)$  tangles described in Example 7.1, then algebras  $\text{End}^*(\alpha_k)$  are both isomorphic to  $\mathcal{A}[-1]$ , and the spaces  $\text{Hom}^*(\alpha_k, \alpha_l)$ ,  $k \neq l$ , both isomorphic to  $\mathcal{A}_0[-1]$ .

**Theorem 2.** *Objects  $E_\alpha = \Phi(\alpha)$ , where  $\alpha$  runs through all crossingless matchings of  $2n$  points, form the set of irreducibles of the heart of the exotic  $t$ -structure on  $\mathcal{D}_{2n}$ . For two crossingless matchings  $\alpha$  and  $\beta$  the space of morphisms  $\text{Hom}^*(E_\alpha, E_\beta)$  is isomorphic to  $\mathcal{A}^{\otimes n_0(\alpha^\vee \circ \beta)} \otimes \mathcal{A}_0^{\otimes n_1(\alpha^\vee \circ \beta)}[-n]$ .*

*Proof.* The key step is a theorem by R. Bezrukavnikov and I. Mirkovic (cf. below) that implies that functors  $G_{2n}^i$  take irreducible objects to irreducible objects. Then objects corresponding to crossingless matchings are irreducible by induction, the base of induction being Example 7.1. And their number  $\binom{2n}{n}$  coincides with the rank of the  $K^0$ -group of  $\mathcal{D}_{2n}$ , hence there could be no more irreducibles. The rest follows from the definitions of  $\mathcal{A}$ ,  $\mathcal{A}_0$  and the preceding argument.

□

The exact statement in [3] of the theorem cited above, with the notation adapted to our particular case, is as follows :

**Theorem 3.** ([3]) *a) For a given  $i$ , there exists a unique  $t$ -structure on  $\mathcal{D}_{2n-2}$ , such that the functor  $G_{2n}^i$  is  $t$ -exact, where the target is equipped with exotic  $t$ -structure.*

*b) If  $G$  is of type  $A_n$  (and in many other cases) the  $t$ -exact functor  $G_{2n}^i$  sends irreducible objects to irreducible ones.*

**Corollary 1.**  $R_{2n}^{2n} \cong \text{id}$ .

*Proof.* By the results of Section 4, for every crossingless matching  $\alpha$  we have  $R_{2n}^{2n} E_\alpha \cong E_\alpha$ . By Theorem 2 the objects  $E_\alpha$  generate  $\mathcal{D}_{2n}$  as a triangulated category, hence the equality of functors holds identically.

□

Note that for all  $\alpha$  we have  $n_0(\alpha^\vee \circ \alpha) = n$ ,  $n_1(\alpha^\vee \circ \alpha) = 0$ . The space  $\text{Hom}(E_\alpha, E_\alpha) \simeq \mathcal{A}^{\otimes n}[-n]$  has the structure of an algebra, and since all  $\alpha$  are permuted by certain braids, all  $E_\alpha$  are permuted by corresponding autoequivalences of  $\mathcal{D}_n$ , and all those algebras are canonically isomorphic. Then the spaces  $\text{Hom}(E_\alpha, E_\beta)$  have the structure of  $\mathcal{A}^{\otimes n}[-n]$ -bimodules  $\mathcal{M}_\alpha^\beta$ .

**Definition 3.** A flat isotopy is an isotopy that does not involve type (I) Reidemeister moves.

**Lemma 9.** Any two isotopic flat links are flat isotopic.

*Proof.* In a flat link diagram no loops intersect, hence for two loops either one lies inside another, or they both lie outside each other. Any flat link is flat isotopic to some link that has a diagram where all 1-loops are concentric circles, and no 0-loop lies inside another loop (construct an isotopy by moving only one loop at a time, and it can be chosen flat). Two links with such diagrams are flat isotopic iff they have equal numbers of 0-loops and of 1-loops, which holds for isotopic links.

□

**Proposition 6.** Any two flat isotopies of the links  $\alpha^\vee \circ \beta$  and  $\gamma^\vee \circ \delta$  give the same map  $\mathcal{M}_\alpha^\beta \rightarrow \mathcal{M}_\gamma^\delta$ .

*Proof.* Let us first consider the special case of a flat isotopy of a flat link to itself, which moves one loop around the origin and leaves all other loops in place. Since  $(R_{2n})^{2n} \cong \text{id}$ , this isotopy yields the identity map on the corresponding graded vector space. Next, all flat isotopies of two flat links are homotopic up to moves of the type considered above. By virtue of the proof of Lemma 4 any homotopy of two affine flat isotopies is equivalent to a composition of linear flat isotopy moves. By the results of [1] if two isotopies of the same links differ only within a simply connected domain (i.e. by a linear isotopy move), they give the same map on the corresponding modules.

□

**Remark.** Consider a sub-2-category in the 2-category of affine framed tangles and their cobordisms, which consists of flat tangles as 1-morphisms and flat isotopies as 2-morphisms. Proposition 6 implies that the weak categorical representation  $\Phi$  we have constructed is actually a strong representation of this sub-2-category.

**Corollary 2.** The isomorphism

$$(55) \quad \mathcal{M}_\alpha^\beta \cong \mathcal{A}^{\otimes n_0(\alpha^\vee \circ \beta)} \otimes \mathcal{A}_0^{\otimes n_1(\alpha^\vee \circ \beta)}[-n]$$

is canonical.

**Theorem 4.** After the canonical identification (55) the composition map  $\mathcal{M}_\alpha^\beta \otimes \mathcal{M}_\beta^\gamma \rightarrow \mathcal{M}_\alpha^\gamma$  becomes a successive application of transformations

- (1) the multiplication maps  $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$  which correspond to merging of two 0-loops;
- (2) the comultiplication maps  $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$  which correspond to splitting of a 0-loop into two 0-loops;

- (3) the action maps  $\mathcal{A} \otimes \mathcal{A}_0 \rightarrow \mathcal{A}_0$  and  $\mathcal{A}_0 \otimes \mathcal{A} \rightarrow \mathcal{A}_0$ , which correspond to merging of a 1-loop and a 0-loop
- (4) the coaction maps  $\mathcal{A}_0 \rightarrow \mathcal{A} \otimes \mathcal{A}_0$  and  $\mathcal{A}_0 \rightarrow \mathcal{A}_0 \otimes \mathcal{A}$  which correspond to splitting of a 1-loop;
- (5) the map  $\mathcal{A}_0 \otimes \mathcal{A}_0 \rightarrow \mathcal{A}$  which corresponds to merging two 1-loops;
- (6) the map  $\mathcal{A} \rightarrow \mathcal{A}_0 \otimes \mathcal{A}_0$  which corresponds to splitting of a 0-loop into two 1-loops

within the tensor product.

Let  $\langle 1, X \rangle$  be the canonical basis of  $\mathcal{A}$ ,  $v, w \in \mathcal{A}_0$ , and let  $\omega(\cdot, \cdot)$  be the canonical symplectic form on  $\mathcal{A}_0$ . Then the maps of Theorem 4 are described as follows:

- (1) merging two separated 0-loops:  $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ :  $1 \otimes * \mapsto *, * \otimes 1 \mapsto *, X \otimes X \mapsto 0$ ;
- (2) merging two 0-loops inside each other, second  $\mathcal{A}$  corresponds to the inner loop:  $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ :  $1 \otimes X \mapsto -X, * \otimes 1 \mapsto *, X \otimes X \mapsto 0$ ;
- (3) splitting a 0-loop into two separated 0-loops:  $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ :  $1 \mapsto 1 \otimes X + X \otimes 1, X \mapsto 0$ ;
- (4) splitting a 0-loop into two 0-loops inside each other, second  $\mathcal{A}$  corresponds to the inner loop:  $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ :  $1 \mapsto -1 \otimes X + X \otimes 1, X \mapsto 0$ ;
- (5) merging a 1-loop with a 0-loop:  $\mathcal{A} \otimes \mathcal{A}_0 \rightarrow \mathcal{A}_0$ :  $1 \otimes * \mapsto *, X \otimes * \mapsto 0$ ;
- (6) splitting a 1-loop into a 1-loop and a 0-loop:  $\mathcal{A}_0 \rightarrow \mathcal{A} \otimes \mathcal{A}_0$ :  $* \mapsto 1 \otimes *$ ;
- (7)  $\mathcal{A}_0 \otimes \mathcal{A}_0 \rightarrow \mathcal{A}$ :  $v \otimes w \mapsto \omega(v, w)X$ ;
- (8)  $\mathcal{A} \rightarrow \mathcal{A}_0 \otimes \mathcal{A}_0$ :  $X \mapsto 0$ ; for  $\omega(v, w) = 1$  we have  $1 \mapsto v \otimes w - w \otimes v$ , and it is easy to see that this vector does not depend on the choice of  $v, w$ .

*Proof.* Consider a sequence of links, first of which is a disjoint union of  $\alpha^\vee \circ \beta$  and  $\beta^\vee \circ \gamma$ , and each subsequent link is obtained from the previous by replacing two symmetric arcs between points  $i$  and  $j$  in the tangle  $\beta \circ \beta^\vee$  by two radial strands at  $i$  and  $j$  so that the tangle remains flat. In the end we arrive at a link which is isotopic to  $\alpha^\vee \circ \gamma$ .

Consider one step in this sequence. Two adjacent members of the sequence are links  $\phi$  and  $\psi$  that differ only inside a simply connected domain  $O$  with 4 marked points  $p_1, p_2, p_3, p_4$  on its boundary, so that  $\phi$  contains arcs  $[p_1, p_2]$  and  $[p_3, p_4]$ , while  $\psi$  contains arcs  $[p_1, p_4]$  and  $[p_2, p_3]$ . By Lemma 9 both  $\phi$  and  $\psi$  are isotopic to collections of circles of normal form. We need a relative analogue of the latter statement for links in the outside of  $O$ , with 4 boundary points  $p_1, p_2, p_3, p_4$ . For this reason, we first repeat the proof of Lemma 9 for the loops that do not intersect  $O$ , and then choose a flat isotopy that minimizes the number of points of inflection outside  $O$ . By Proposition 6 the map that we obtain between the corresponding spaces does not depend on the choice of an isotopy.

We deduce that the map between the spaces corresponding to  $\phi$  and  $\psi$  is canonically isomorphic to a map between spaces corresponding to two links from a finite list. Thus, the map may be calculated directly using the results of two examples in the next subsection. The possible configurations of links to which we reduce our calculations is as follows.

- (1) Assume that the link  $\phi$  connects points  $p_1$  and  $p_2$ , and  $p_3$  and  $p_4$  outside  $O$ . Here we are in the situation of the Example 7.1 below. This case gives us the map 1 of multiplication in  $\mathcal{A}$ , the maps 5 of left and right actions of  $\mathcal{A}$  on  $\mathcal{A}_0$ , and the map 7.



Assume that the link  $\phi$  connects points  $p_1$  and  $p_4$ , and  $p_3$  and  $p_2$  outside  $O$ . We adopt the notation of Example 7.2 below.

- (2)  $\phi \sim (\beta_1)^\vee \circ g_4^2 \circ f_4^2 \circ \beta_1$ . This gives us the comultiplication map 3.
- (3)  $\phi \sim (\beta_2)^\vee \circ g_4^2 \circ f_4^2 \circ \beta_1$ . This gives us the coaction map 6.
- (4)  $\phi \sim (\beta_2)^\vee \circ g_4^2 \circ f_4^2 \circ \beta_3$ . This gives us the map 8.

Other options are impossible, because  $\phi$  is a flat link, and its intersection with a simply connected domain  $O$  consists of two arcs only.

The case of two 0-loops inside each other is treated in detail in [1]. It differs from the case of two separate 0-loops by the following argument. Consider the composition of a Reidemeister II and two Reidemeister I isotopies:

$$f_n^i g_n^i \rightarrow f_n^i t_n^i(1) t_n^i(2) g_n^i \rightarrow f_n^i g_n^i.$$

It maps the basis  $1, X$  to  $1, -X$ . Note that this map does not depend on the order of  $t_n^i(1)$  and  $t_n^i(2)$  in the middle, hence proving the independence of the morphism (55) on the way of pulling the inner loop outside. Denote this map by  $\iota$ . Then the multiplication map for two embedded 0-loops takes form

$$(56) \quad \mathcal{A} \otimes \mathcal{A} \xrightarrow{\iota \otimes id} \mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A} \xrightarrow{\iota^{-1}} \mathcal{A}$$

which proves the point.

□

## 7. EXAMPLES.

**7.1. The case of  $n = 1$ .** When  $n = 1$ , we have  $\mathcal{B}_{Z_1} \cong \mathbb{P}^1 = \mathbb{P}(V)$ ,  $U_{Z_1} \cong T^\vee \mathbb{P}^1$ ,  $\mathcal{E}_1 \cong \mathcal{O}(-1)$ ,  $\mathcal{E}_2 \cong T_{\mathbb{P}^1}(-1)$ . The divisor  $D_1$  is simply  $\mathbb{P}^1$ , and  $\mathcal{O}(D_1) \cong T_{\mathbb{P}^1}^\vee$ . The functor  $G_2^1 : \mathcal{D}_0 \cong D^b(pt) \cong Vect \rightarrow \mathcal{D}_1$  is given by  $\mathbb{C} \mapsto \iota_*(\mathcal{O}(-1) \otimes \pi^* \mathbb{C}) \cong \iota_* \mathcal{O}(-1)$ . The twist  $T_2^1(1)$  is the twist in the spherical object  $\iota_* \mathcal{O}(-1)$ , which is, the cone of the morphism of functors  $\iota_* \mathcal{O}(-1) \otimes \text{Hom}(\iota_* \mathcal{O}(-1), \cdot) \rightarrow id$ . It sends  $\iota_* \mathcal{O}(-1)$  to  $\iota_* \mathcal{O}(-1)[-1]$ ,  $\iota_* \mathcal{O}$  to  $\iota_* T_{\mathbb{P}^1}^\vee[1]$ . Then the functor  $R_2 = T_2^1(2) \otimes \mathcal{E}_2^\vee \otimes L$  (where  $L = \sqrt{V} \wedge V$ ) sends  $\iota_* \mathcal{O}(-1)$  to  $\iota_* T_{\mathbb{P}^1}^\vee \otimes L[1]$ , and  $\iota_* T_{\mathbb{P}^1}^\vee \otimes L[1]$  to  $\iota_*(\mathcal{O}(-1) \otimes T_{\mathbb{P}^1}(-2)) \otimes L^{\otimes 2}$ . Note that  $T_{\mathbb{P}^1}(-2) \cong \mathcal{E}_1 \otimes \mathcal{E}_2 \cong \mathcal{O} \otimes \Lambda^2 V$ , hence  $\iota_*(\mathcal{O}(-1) \otimes T_{\mathbb{P}^1}(-2)) \otimes L^{\otimes 2} \cong \iota_* \mathcal{O}(-1)$ , quite expectedly.

The objects  $\iota_* \mathcal{O}(-1)$  and  $\iota_* T_{\mathbb{P}^1}^\vee \otimes L[1]$  are the irreducibles in the exotic  $t$ -structure. Denote them  $\alpha_1$  and  $\alpha_2$ . The spaces of morphisms between them are as follows:

$$\text{Hom}^0(\alpha_i, \alpha_i) \cong H_{\mathbb{P}^1}^0(\mathcal{O}) \cong \mathbb{C}; \quad \text{Hom}^1(\alpha_i, \alpha_i) = 0; \quad \text{Hom}^2(\alpha_i, \alpha_i) \cong H_{\mathbb{P}^1}^1(\mathcal{O}(-2)) \cong \mathbb{C};$$

$$\text{Hom}^0(\alpha_1, \alpha_2) = 0; \quad \text{Hom}^1(\alpha_1, \alpha_2) \cong H_{\mathbb{P}^1}^1(T_{\mathbb{P}^1}^\vee(-1) \otimes L) \cong V \otimes \sqrt{V \wedge V}^\vee \cong \mathbb{C}^2;$$

$$\text{Hom}^2(\alpha_1, \alpha_2) = 0.$$

There is a canonically defined symplectic form  $\omega$  on  $\text{Hom}^1(\alpha_1, \alpha_2)$ , and a canonical isomorphism  $\text{Hom}^1(\alpha_1, \alpha_2) \cong \text{Hom}^1(\alpha_2, \alpha_1)$ . Let us denote the space  $\text{Hom}^*(\alpha_i, \alpha_i)$  by  $A$ , its basis by  $\langle 1, X \rangle$ , and let  $v, w$  be any elements of  $A_1 := \text{Hom}^1(\alpha_1, \alpha_2)$ . The composition rules are given as follows:

$$(57) \quad A \otimes A \rightarrow A : \quad 1 \otimes * \mapsto *, \quad * \otimes 1 \mapsto *, \quad X \otimes X \mapsto 0;$$

(hence  $A$  is a commutative algebra, and  $A_1$  a bimodule over it)

$$(58) \quad A \otimes A_1 \rightarrow A_1 : \quad 1 \otimes v \mapsto v, \quad X \otimes v \mapsto 0;$$

(the right action is the same)

$$(59) \quad A_1 \otimes A_1 \rightarrow A : \quad v \otimes w \mapsto \omega(v, w)X.$$

In fact,  $A$  and  $A_1$  together form the wedge algebra of some  $W = \mathbb{C}^2$ :  $A \cong \Lambda^0(W) \oplus \Lambda^2(W)$ ,  $A_1 \cong \Lambda^1(W)$ .

**7.2. The case of  $n = 2$ .** Denote  $Z = Z_2$ , and let  $V$  be the standard representation of  $\mathfrak{g} = \mathfrak{sl}_4$ . The Grothendieck-Springer condition turns then into

$$(60) \quad \mathcal{B}_Z \cong \{(0 = V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 = V) | ZV_i \subset V_{i-1}, i = 1, \dots, 4\}.$$

The spaces of partial flags are

$$(61) \quad \mathcal{P}_{1,Z} \cong \{(0 \subset V_2 \subset V_3 \subset V) | ZV_2 = 0, ZV_3 \subset V_2, ZV \subset V_3\} \cong \mathbb{P}(V/\text{Ker } Z) \cong \mathbb{P}^1;$$

$$(62) \quad \mathcal{P}_{2,Z} \cong \{(0 \subset V_1 \subset V_3 \subset V) | ZV_1 = 0, ZV_3 = V_1, ZV \subset V_3\} \cong \mathbb{P}(V/\text{Ker } Z) \cong \mathbb{P}^1;$$

$$(63) \quad \mathcal{P}_{3,Z} \cong \{(0 \subset V_1 \subset V_2 \subset V) | ZV_1 = 0, ZV_2 \subset V_1, ZV = V_2\} \cong \mathbb{P}(\text{Ker } Z) \cong \mathbb{P}^1.$$

The space  $\mathcal{B}_Z$  itself has two irreducible components: the first is characterized by  $ZV_2 = 0$ , hence  $V_2 = \text{Ker } Z$ ,  $V_1 \in \mathbb{P}(\text{Ker } Z)$ , and  $V_3 \in \mathbb{P}(V/\text{Ker } Z)$ . It is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ . The second is characterized by  $ZV_2 \neq 0$ , hence  $ZV_3 \subset \text{Ker } Z \cap V_2 = V_1$ , hence  $ZV_3 = V_1$ . This component is fibered over  $\mathcal{P}_{2,Z}$  with fiber  $\mathbb{P}(V_3/V_1) \cong \mathbb{P}^1$ . To find the fibration, notice that the restrictions of the tautological vector bundles  $\mathcal{V}_1$  and  $\mathcal{V}_3$  on  $\mathcal{P}_{2,Z}$  are  $\mathcal{O}(-1)$  and  $\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1)$  respectively, and the map  $\mathcal{V}_1 \rightarrow \mathcal{V}_3$  is the canonical embedding  $\mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus 2}$ . Then the projectivization of  $\mathcal{V}_3/\mathcal{V}_1$  is the Hirzebruch surface  $\mathbb{F}_2$ . The intersection of two components is

$$(64) \quad \{(0 \subset V_1 \subset V_2 \subset V_3 \subset V) | ZV_1 = 0, ZV_2 = 0, ZV_3 = V_1, ZV \subset V_3\}$$

which is projected isomorphically onto all three  $\mathcal{P}_{k,Z}$ , whence isomorphic to  $\mathbb{P}^1$ . It is the diagonal in  $\mathbb{P}^1 \times \mathbb{P}^1$ , and the exceptional line  $\mathbb{P}_{exc}^1$  in  $\mathbb{F}_2$ .

Denote by  $p_1$  and  $p_3$  the projections of  $\mathbb{P}^1 \times \mathbb{P}^1$  onto  $\mathcal{P}_{1,Z}$  and  $\mathcal{P}_{3,Z}$  respectively, and denote by  $p_2$  the projection of  $\mathbb{F}_2$  onto  $\mathcal{P}_{2,Z}$ . Then the tautological line bundles  $\mathcal{E}_i$  are as follows:

$$\begin{aligned} \mathcal{E}_1|_{\mathbb{P}^1 \times \mathbb{P}^1} &\cong p_3^* \mathcal{O}(-1); & \mathcal{E}_2|_{\mathbb{P}^1 \times \mathbb{P}^1} &\cong p_3^* T_{\mathbb{P}^1}(-1); \\ \mathcal{E}_3|_{\mathbb{P}^1 \times \mathbb{P}^1} &\cong p_1^* \mathcal{O}(-1); & \mathcal{E}_4|_{\mathbb{P}^1 \times \mathbb{P}^1} &\cong p_1^* T_{\mathbb{P}^1}(-1); \\ \mathcal{E}_1|_{\mathbb{F}_2} &\cong p_2^* \mathcal{O}(-1); & \mathcal{E}_2|_{\mathbb{F}_2} &\cong \mathcal{O}_{p_2}(-1); & \mathcal{E}_3|_{\mathbb{F}_2} &\cong T_{p_2}(-1); & \mathcal{E}_4|_{\mathbb{F}_2} &\cong p_2^* T_{\mathbb{P}^1}(-1). \end{aligned}$$

Here  $\mathcal{O}_{p_2}(-1)$  means the tautological bundle on  $\mathbb{F}_2$  viewed as a projectivization of  $\mathcal{O}(-1) \oplus T_{\mathbb{P}^1}(-1)$  on  $\mathbb{P}^1$ , and  $T_{p_2}$  is the fiberwise tangent bundle. Note that  $\mathcal{O}_{p_2}(-1)|_{\mathbb{P}_{exc}^1} \cong T_{\mathbb{P}^1}(-1)$ .

Denote by  $i$  and  $j$  the maps  $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow U_Z$  and  $\mathbb{F}_2 \rightarrow U_Z$  respectively. The functors  $G_4^k = i_{k*}(\mathcal{E}_k \otimes \pi_k^*)$ ,  $k = 1, 2, 3$ , map the pair  $\iota_* \mathcal{O}(-1)$ ,  $\iota_* T_{\mathbb{P}^1}^\vee \otimes L[1]$  of irreducibles for  $n = 1$  to the pairs

$$(65) \quad \beta_1 = i_*(p_1^* \mathcal{O}(-1) \otimes p_3^* \mathcal{O}(-1)), \quad \beta_2 = i_*(p_1^* T_{\mathbb{P}^1}^\vee \otimes p_3^* \mathcal{O}(-1) \otimes L)[1];$$

$$(66) \quad \beta_4 = j_*(\mathcal{O}_{p_2}(-1) \otimes p_2^* \mathcal{O}(-1)), \quad \beta_5 = j_*(\mathcal{O}_{p_2}(-1) \otimes p_2^* T_{\mathbb{P}^1}^\vee \otimes L)[1];$$

$$(67) \quad \beta_1 = i_*(p_1^* \mathcal{O}(-1) \otimes p_3^* \mathcal{O}(-1)), \quad \beta_3 = i_*(p_1^* \mathcal{O}(-1) \otimes p_3^* T_{\mathbb{P}^1}^\vee \otimes L)[1].$$

These are five of the six irreducibles for  $n = 2$ . The sixth is obtained as  $\beta_6 \cong R_4\beta_2$ , or, alternatively, by building a resolution of  $\mathcal{O}_{pt}$  by direct sums of irreducible objects. The latter method is suggested by the fact that  $\mathcal{O}_{pt}$  lies in the heart of the exotic  $t$ -structure.

We can carry out one of the possible computations of  $\beta_6$ . Namely,

$$(68) \quad \beta_6 \cong T_4^1(1)\mathcal{S}_4T_4^2(1)\beta_3[1] \cong \{G_4^1F_4^1\mathcal{S}_4T_4^2(1)\beta_3 \rightarrow \mathcal{S}_4T_4^2(1)\beta_3[1]\} \cong \{\beta_1 \rightarrow \mathcal{S}_4T_4^2(1)\beta_3[1]\}.$$

Compute

$$(69) \quad T_4^2(1)\beta_3 \cong \{G_4^2F_4^2G_4^3\alpha_2[-1] \rightarrow G_4^3\alpha_2\} \cong \{\beta_5[-1] \rightarrow \beta_3\}.$$

We obtain  $\mathcal{S}_4T_4^2(1)\beta_3[1]$  as a line bundle  $\mathcal{F}[2]$  on  $B_Z$ , where the restriction of  $\mathcal{F}$  to  $\mathbb{P}^1 \times \mathbb{P}^1$  is  $p_1^*T_{\mathbb{P}^1}^\vee \otimes p_3^*T_{\mathbb{P}^1}^\vee \otimes L^{\otimes 2} \otimes \mathcal{O}(\Delta)$  and the restriction of  $\mathcal{F}$  to  $\mathbb{F}_2$  is  $\mathcal{O}_{p_2}(-1) \otimes p_2^*((T_{\mathbb{P}^1}^\vee)^{\otimes 2}(1)) \otimes L^{\otimes 2}$ .

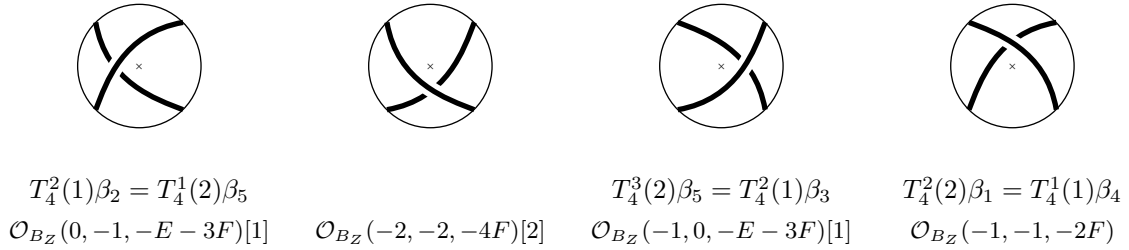
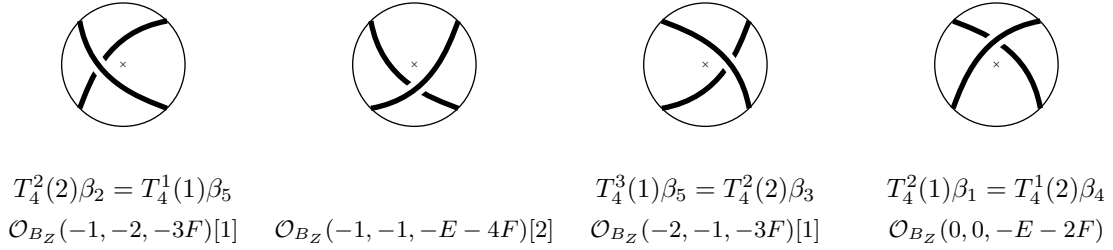
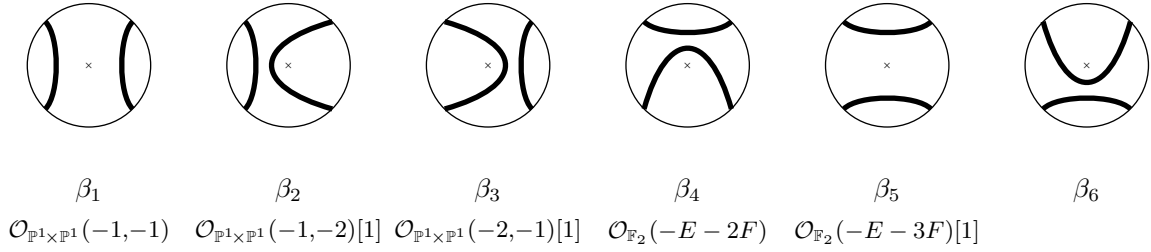
If we drop control of the scalar factor necessary for defining the multiplication on Hom spaces, and only consider our objects up to isomorphism (which is a scalar, since for every  $E_\beta$  holds  $\text{Hom}^0(E_\beta, E_\beta) \simeq \mathbb{C}$ ), we may write  $\beta_6$  as the cone

$$(70) \quad \beta_6 \simeq \{i_*\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \rightarrow \mathcal{O}_{B_Z}(-1, -1, -E - 3F)[2]\},$$

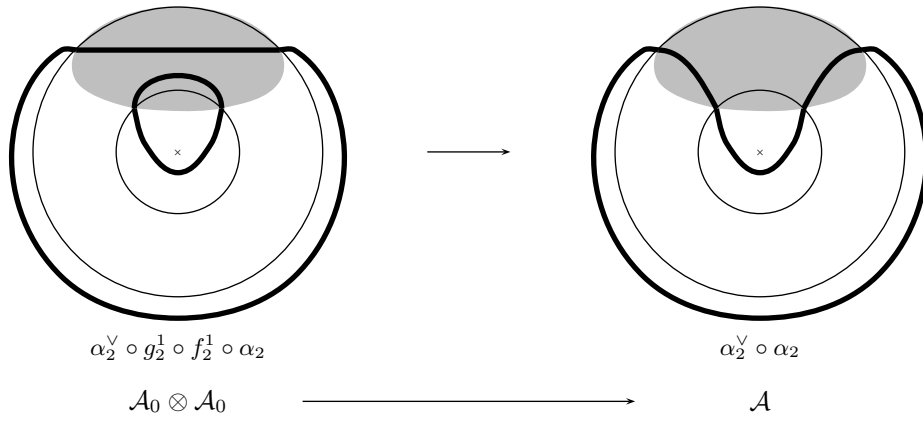
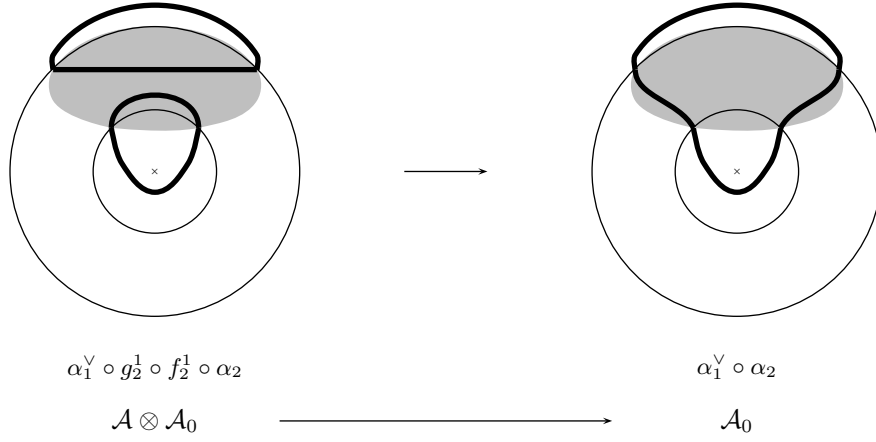
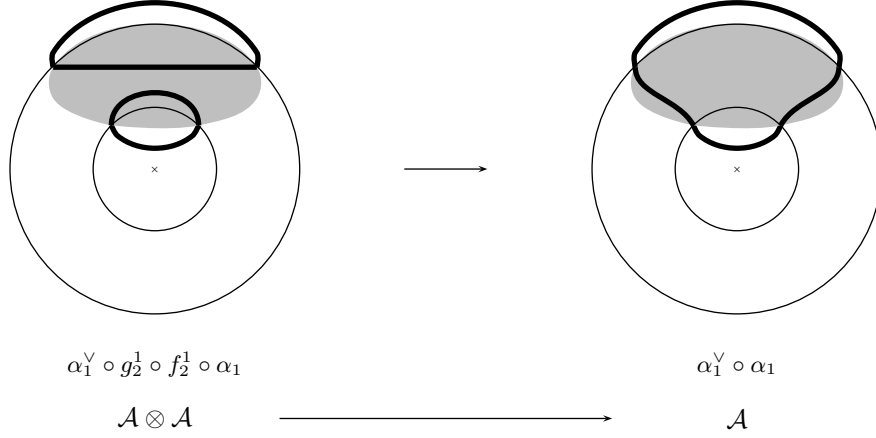
where  $\mathcal{O}_{B_Z}(-1, -1, -E - 3F)$  is a line bundle on  $B_Z = \mathbb{P}^1 \times \mathbb{P}^1 \cup \mathbb{F}_2$  such that its restriction on  $\mathbb{P}^1 \times \mathbb{P}^1$  is isomorphic to  $\mathcal{O}(-1, -1)$ , and its restriction on  $\mathbb{F}_2$  is isomorphic to  $\mathcal{O}_{\mathbb{F}_2}(-E - 3F)$ , where  $E$  is the divisor of the exceptional line on  $\mathbb{F}^2$ , and  $F$  is the divisor of the fiber of  $p_2 : \mathbb{F}^2 \rightarrow \mathbb{P}^1$ .

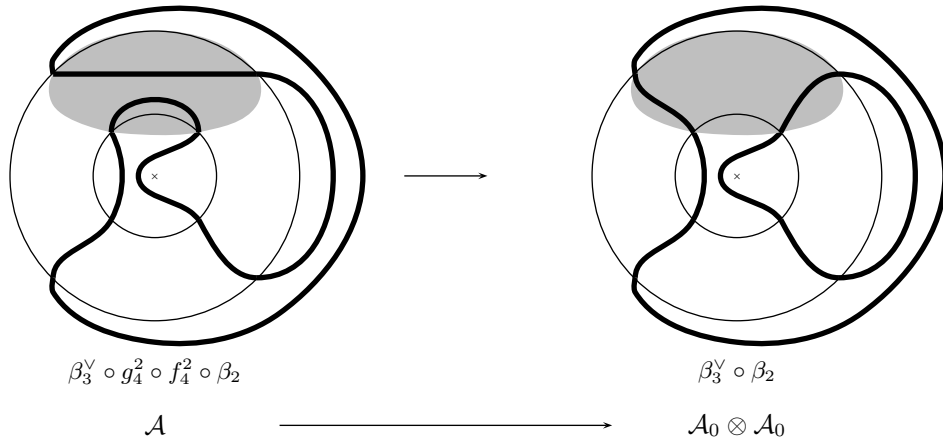
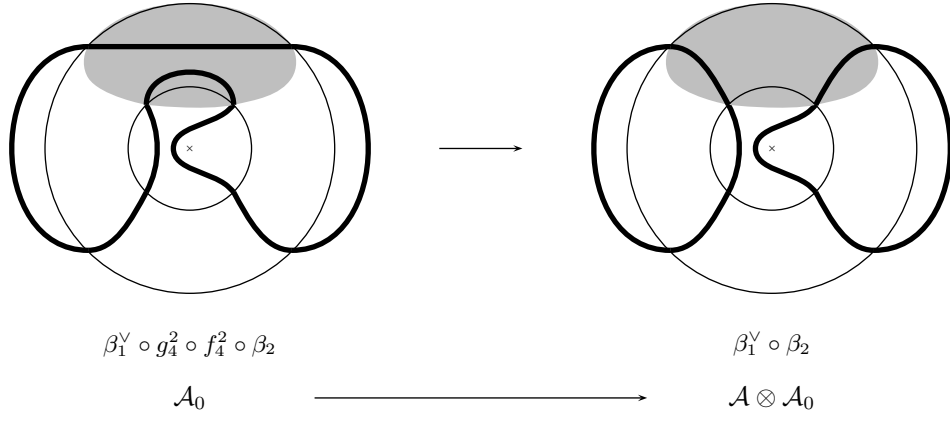
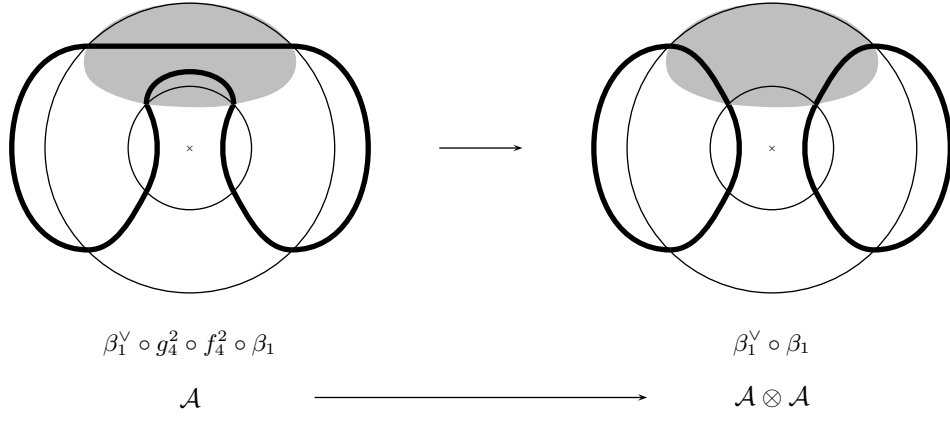
## 8. APPENDIX

**8.1. Some diagrams for  $n = 2$ .** In this section we present a list of several isomorphism classes of objects that correspond to certain tangle diagrams. For simplicity, we write our representatives as tensor products of classes in the Picard group of  $U_Z = U_{Z_2}$  and structure sheaves of certain subvarieties, namely  $B_Z \subset U_Z$  and its components, which are isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $\mathbb{F}^2$ . The Picard group elements are written in the form  $\mathcal{O}(a, b, cE + dF)$ , where  $a$  and  $b$  denote intersection numbers with the fibers of  $p_1$  and  $p_3$  respectively (note that it means that  $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b) \simeq p_1^* \mathcal{O}(b) \otimes p_3^* \mathcal{O}(a)$ ), and  $cE + dF$  is a divisor on  $\mathbb{F}^2$ , where  $E$  is the exceptional line, and  $F$  is the fiber of  $p_2$ .



**8.2. The maps of Theorem 4.** Here we present six maps of Theorem 4 that correspond to splitting or merging of two loops. Shaded areas indicate the region where a tangle  $g_n^i \circ f_n^i$  is replaced by  $id$  (this region was denoted by  $O$  in the proof of Theorem 4).





## REFERENCES

- [1] R. Anno, Spherical functors; [arXiv:0711.4409](#).
- [2] R. Bezrukavnikov, Noncommutative counterparts of the Springer resolution; [math.RT/0604445](#).
- [3] R. Bezrukavnikov, private communication.
- [4] S. Cautis, J. Kamnitzer, Knot homology via derived categories of coherent sheaves I,  $sl(2)$  case; [math.AG/0701194](#).
- [5] R.P. Horja, Derived Category Automorphisms from Mirror Symmetry, *Duke Math. J.* **127** (2005), 1-34; [math.AG/0103231](#).
- [6] M. Jacobsson, An invariant of link cobordisms from Khovanov homology, [math.QA/0206303](#).
- [7] C. Kassel, Quantum Groups, Springer-Verlag, 1995.
- [8] M. Khovanov, *A functor-valued invariant of tangles*, [math.QA/0103190](#).
- [9] M. Khovanov and R. Thomas, Braid cobordisms, triangulated categories, and flag varieties; [math.QA/0609335](#).
- [10] P. Seidel, private communication.
- [11] C. Stroppel, Perverse sheaves on Grassmannians, Springer fibres and Khovanov homology; [math.RT/0608234](#).
- [12] C. Stroppel, Categorification of the Temperley-Lieb category, tangles, and cobordisms via projective functors, *Duke Math. J.*, **126** (2005) no. 3, 547–596.
- [13] C. Stroppel, private communication.
- [14] C. Stroppel, B. Webster, 2-block Springer fibers: convolution algebras, coherent sheaves, and embedded TQFT; [math.RT/0802.1943](#).

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